Complex behavior in one-dimensional sandpile models

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We give some examples to illustrate that scale invariance may not be a manifestation of complex behavior in one-dimensional sandpile models. The multiscaling statistical properties and the existence of intrinsic length scales observed in the local limited one-dimensional model reflects a certain level of complexity. The local, limited, and limited to no traps model presents scale invariance due to the inhomogeneous way of perturbing the lattice. It behaves, however, as the trivial one-dimensional version of the Bak, Tang, and Wisenfeld [Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988)] model. A nonlocal limited model presents scaling statistical properties and displays the same level of complexity as the nontrivial two-dimensional models. [S1063-651X(97)02102-8]

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I. INTRODUCTION

Sandpile models have been used by Bak, Tang, and Wiesenfeld [1] to introduce the concept of self-organized criticality (SOC). Many systems out of equilibrium do not require the tuning of an external parameter to be driven to a (self-organized) scale-invariant stationary state. It is generally accepted [2–4] that SOC is produced by the interplay between a very slow driving dynamics (sand is added to a sandpile at a constant slow rate) and a short relaxation beyond a certain threshold parameter (eventually, avalanches of sand will begin). In many investigations of extended dissipative dynamical systems, there is an identification between SOC and the occurrence of some sort of scaling invariance.

In the sandpile models ''complexity'' usually means the existence of avalanches of all sizes [5,6], regardless of the way of implementing the rules of the dynamics. There are several studies of sandpile models characterized by a ''complex behavior'' associated with power laws and scaling invariance (SI). An extended system has a ''complex behavior'' if it presents nontrivial dynamics out of simple dynamical rules (the ''degree of complexity'' may be related to the ability of understanding these global dynamical properties). In this paper, we revisit some of the one-dimensional sandpile models with the purpose of investigating the connections between SI and ''complexity.'' We show that in some cases the power laws observed in the statistical analysis of these one-dimensional models are produced by avalanches of sizes which do depend on the way the system is perturbed. In particular, we consider the mechanisms of the avalanches in a class of ''limited'' sandpile models introduced by Kadanoff *et al.* [7].

To be more specific, we present a comparative analysis of four one-dimensional limited sandpile models: (i) The simple one-dimensional version of the original model proposed by Bak, Tang, and Wiesenfeld [1] (which we call the BTW1 model); (ii) The local, limited (LL) model proposed by Kadanoff and collaborators [7]; (iii) The nonlocal limited (NLL) model [7]; (iv) The local limited and limited to no traps (LLL) model proposed by Chhabra *et al.* [8]. We propose a kind of ''complexity hierarchy'' in the context of these models. The BTW1 model displays just a trivial behavior. In the LL model, the presence of SI corresponds to a ''complex behavior,'' but there are intrinsic length scales in addition to the size of the system. The LLL model presents SI but, unlike the original two-dimensional BTW model, it does not exhibit any ''complex behavior.'' The NLL model is closer to the two-dimensional models. It does present complex behavior, which is not associated with any intrinsic length scales.

This paper is organized as follows. In Sec. II, we review the dynamical rules that are used to define the limited one-dimensional models, and describe the basic ideas of the statistical analysis. In Sec. III, we consider the behavior of these models, and describe the mechanisms that are responsible for SI. In Sec. IV, we present some conclusions and raise a few questions about SOC, SI, and ''complexity.''

II. ONE-DIMENSIONAL LIMITED MODELS

Sandpile models on a lattice are continuously perturbed by the local addition of a certain number of ''grains of sand.'' An integer variable (for instance, the height of the sandpile) is associated with each site of the lattice. At a time t, the site to be perturbed may be chosen randomly or may be determined beforehand. If the height of any site (or else the slope, that is, the difference between the heights of two adjacent sites) exceeds an integer threshold value, a stability criterion is violated and an avalanche begins. Sand is then redistributed to other sites, according to a variety of dynamical rules, obeying a local law of conservation. The grains of sand that are moved to other sites may turn them unstable and give rise to additional falls of sand. At the moment all heights (or slopes) are below or equal the critical threshold values, this relaxation process stops, and the continuous addition of sand is resumed. In those systems there are two well separated time scales, which are an essential characteristic of SOC [3]. At the driven time scale, the system is perturbed; at the relaxation time scale, the avalanches occur. With at least one open boundary, sand will eventually fall off the system. A statistically stationary state is reached when, on the average, the same number of grains is flowing into and out of the system, so that there is a global conservation of sand.

We now describe the rules of the limited models. As in the work of Kadanoff and collaborators [7], the analysis is based on the distribution function of the number of drops (d), that is, the number of grains that drop off the open edge of the lattice, and the distribution function of the number of flips (f), that is, the total number of flipping events (which gauges the avalanche size). We call them drop and flip distributions, respectively, and use a statistical procedure to smooth all data [9].

Consider a one-dimensional lattice of L sites, and an integer height variable h_k representing the number of grains at site $k=1,\ldots,L$. At a time t, suppose that n_p grains are added to a randomly chosen site i,

$$h_i \rightarrow h_i + n_p . \tag{1}$$

In the one-dimensional models discussed in this paper, the stability criterion is based on the slope, $S_i = h_i - h_{i-1}$. If $S_i > S_c$, where S_c is a critical threshold, stability is violated and $n_f(i)$ grains of sand topple from site i,

$$h_i \rightarrow h_i - n_f(i). \tag{2}$$

In the limited models [7], $n_f(i)$ is a constant integer value, $n_f(i) = n$. All the limited models that we consider have one open and one closed boundary. So, $h_0 = 0$, for the open boundary, and $h_{L+1} = h_L$, for the closed ending.

In the local limited (LL) model, if site i becomes unstable, n grains of sand topple from site i to the nearest-neighbor site $i-1$. We have

$$h_i \rightarrow h_i - n, \tag{3}$$

and

$$h_{i-1} \rightarrow h_{i-1} + n, \tag{4}$$

where n is an integer. Once a site is perturbed, the criterion of stability is tested along the whole lattice. If it is violated at a site k, n grains topple from k to the nearest neighbor $k-1$.

The BTW1 model is a particular case of the LL model for $n_p = n$ (the number of grains added to a given site during the process of perturbation is the same as the number of grains that topple from a site that becomes unstable). This simple rule is responsible for the well known trivial behavior of this model. In the original LL model introduced by Kadanoff and collaborators [7], we take $n_p = 1$ and $n = 2$. For $n_p < n$, avalanches propagating towards the closed edge (which we call backward avalanches) are the source of a nontrivial behavior. On the other hand, for $n_p = kn$, where k is an integer, computer simulations indicate just a trivial behavior.

In the nonlocal limited model (NLL), if the stability criterion is violated, n grains of sand fall from site i to sites $i-j$, with $j = 1,2,\ldots,n$. The dynamical rules are given by

$$h_i \rightarrow h_i - n, \tag{5}$$

and

$$h_{i-1} \rightarrow h_{i-1} + 1, \ldots, h_{i-n} \rightarrow h_{i-n} + 1, \tag{6}$$

TABLE I. The dynamical rules of the LLL model ($n_p = 1$, $n = 2$, and $S_c = 2$).

ϵ_{i+1}	ϵ_i	Perturbation	Drops	Flips	ϵ_{i+1}	ϵ_i
0	0	forbidden	-	-	0	0
1	0	permitted	0	0	0	1
0	1	permitted	2	i	1	0
1	1	permitted	$2(L-i+1)$	$\Sigma_{j=i}^{L} j$	0	0

where n is an integer. This model was used by one of us to introduce the effects of inertia that should be relevant in sandpiles [10].

It is convenient to express the relaxation rules in terms of a new variable, $\epsilon_k = S_k - 1$. Suppose, for example, that $n_p = 1$, $n = 2$, and $S_c = 2$. In this example, $\epsilon_k \leqslant 1$, $\forall k$, for both the LL and the NLL models. When the site i is perturbed, there are two possibilities: if $\epsilon_i < 1$, there is a simple addition of a grain of sand in this site $(f=0)$. However, if $\epsilon_i = 1$, an avalanche is triggered $(f \neq 0)$. The boundary conditions are written as $\epsilon_1 = h_1 - 1$ and $\epsilon_{L+1} = -1$.

In the LL model there are special sites on the lattice which stop an avalanche process. They were named traps [8] or troughs [11] in the literature. They are defined as the sites where $\epsilon_t < S_c - n$. In the previous example, the traps are sites for which $\epsilon_t < 0$.

The local, limited, and limited to no traps (LLL) model [8] is a variation of the LL model where the perturbations at the sites which might give rise to a trap are forbidden. It is thus forbidden to perturb a site k where $\epsilon_k = \epsilon_{k+1} = S_c - n$. The result is a model with no traps. When an avalanche is triggered, it propagates until the open edge. If it eventually propagates backwards, it goes towards the closed edge, reflects there, and finally ends at the open edge, with grains of sand falling off the pile. In the previous example ($n_p = 1$, $n = 2$, and $S_c = 2$), the forbidden sites are associated with $\epsilon_k = \epsilon_{k+1} = 0$, the possible values of ϵ_k are 0 or 1, and we redefine the closed boundary conditions, $\epsilon_{L+1} = 1$. Now, suppose that a site i is perturbed. If the stability criterion is violated, the dynamical rules are given in Table I (with an exception, at the closed boundary, as shown in Table II). From Table I, we can also see that the LLL model corresponds to a diffusion-limited annihilation reaction, $A + A \rightarrow 0$, as pointed out by Krug [12]. The variable ϵ_i represents the presence $(\epsilon_i = 1)$ or absence $(\epsilon_i = 0)$ of a particle A at site i. The situations described at the second $[(\epsilon_i, \epsilon_{i+1}) = (0,1) \rightarrow (1,0)]$ and the third lines $[(\epsilon_i, \epsilon_{i+1}) = (1,0) \rightarrow (0,1)]$ of Table I correspond to the diffusive motion of particles, while the pairwise annihilation is described by the forth line $[(\epsilon_i, \epsilon_{i+1}) = (1,1) \rightarrow (0,0)]$.

TABLE II. The dynamical rules of the site L of the LLL model ($n_p = 1$, $n = 2$, and $S_c = 2$).

ϵ_{L+1}	ϵ_L	Perturbation	Drops	Flips	ϵ_{L+1}	ϵ_L
1	0	never forbidden	0	0	1	1
1	1	never forbidden	2	L	1	0

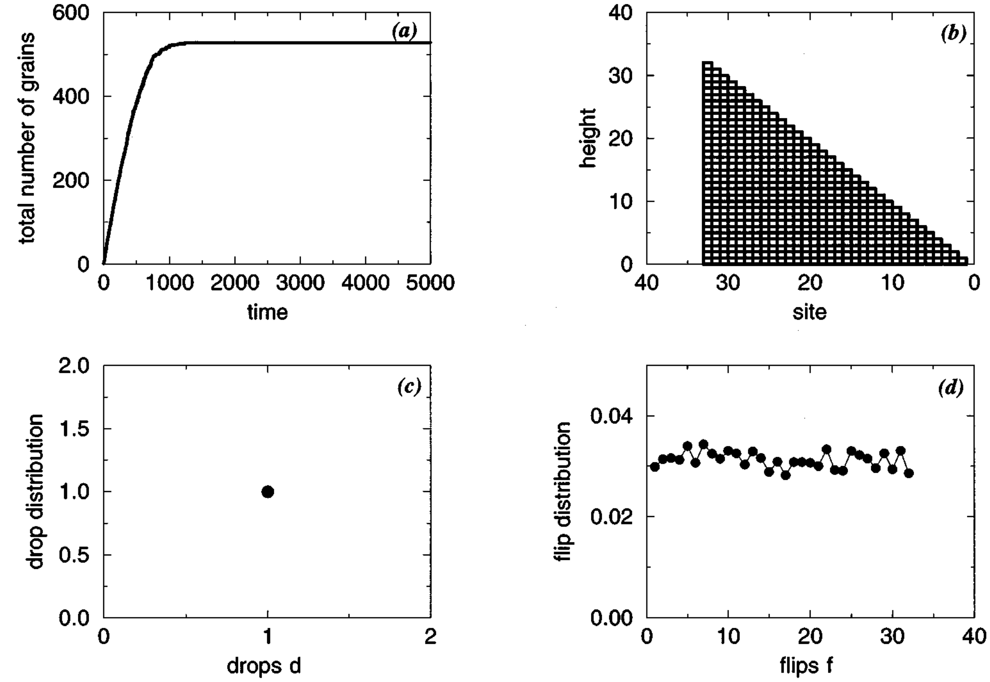

FIG. 1. Statistical analysis of the one-dimensional version of the BTW model with $n_p = n = 1$, $L = 32$, and $S_c = 1$. (a) Total number of grains on the lattice vs time; (b) Configuration of the stationary state; (c) Drop distribution (ratio between the number of events with d drops and the number of events with avalanches) in the stationary state; (d) Flip distribution (ratio between the number of events with f flips and the number of events with avalanches) in the stationary state.

III. AVALANCHE DYNAMICS OF THE LIMITED MODELS

A. The LL model

As we mentioned before, the BTW1 model is a particular case of the LL model for $n_p = n$. In terms of the variables $\{\epsilon_k\}$, the stationary state [see Fig. 1(a)] is a sequence of zeros [see Fig. 1(b)]. Every time a site is perturbed, an avalanche is triggered, and the associated number of drops is always $d = n$ [see Fig. 1(c)]. The flip distribution reveals the homogeneous character of the random perturbations [see Fig. 1(d)]. If site i is perturbed, there is only a corresponding number $f = i$ of flips. As $n_p = n$, the behavior of the BTW1 model is trivial.

For $n > n_p$, the LL model exhibits a complex behavior that has been investigated by many authors [8,11,13,14]. It has some important features: (i) The existence of backward avalanches, that is, avalanches that propagate forwards and also backwards along the lattice. In the BTW1 model, there are only forward avalanches; (ii) The presence of traps, that is, the existence of sites that work as boundaries to limit the regions where the avalanches occur (even after the stationary state is reached). In the BTW1 model, as the traps exist only in the transient regime, an avalanche in the stationary state stops in the boundary only.

In the LL model, a backward avalanche is triggered for $\epsilon_i = \epsilon_{i+1} = 1$ [see Fig. 2(a)]. For $\epsilon_i = 1$ and $\epsilon_{i+1} < 1$, only

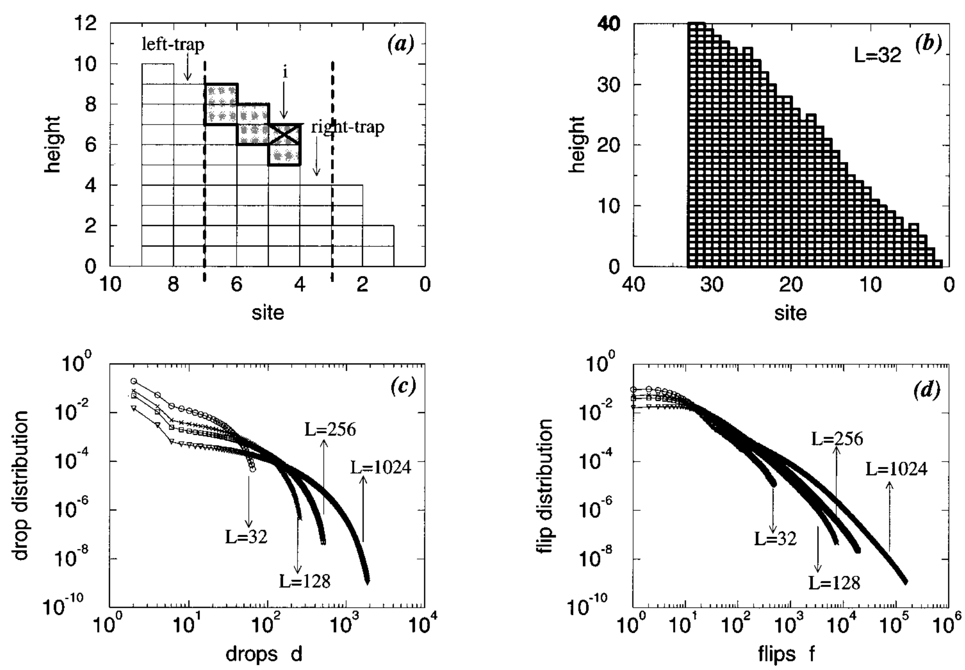

FIG. 2. Statistical analysis of the LL model with $n_p = 1$, $n = 2$, and $S_c = 2$. (a) An example of configuration in which a backward avalanche is triggered and stops at the traps; (b) Configuration of one of the possible stationary states; (c) Drop distribution for different lattice sizes; (d) Flip distribution for different lattice sizes.

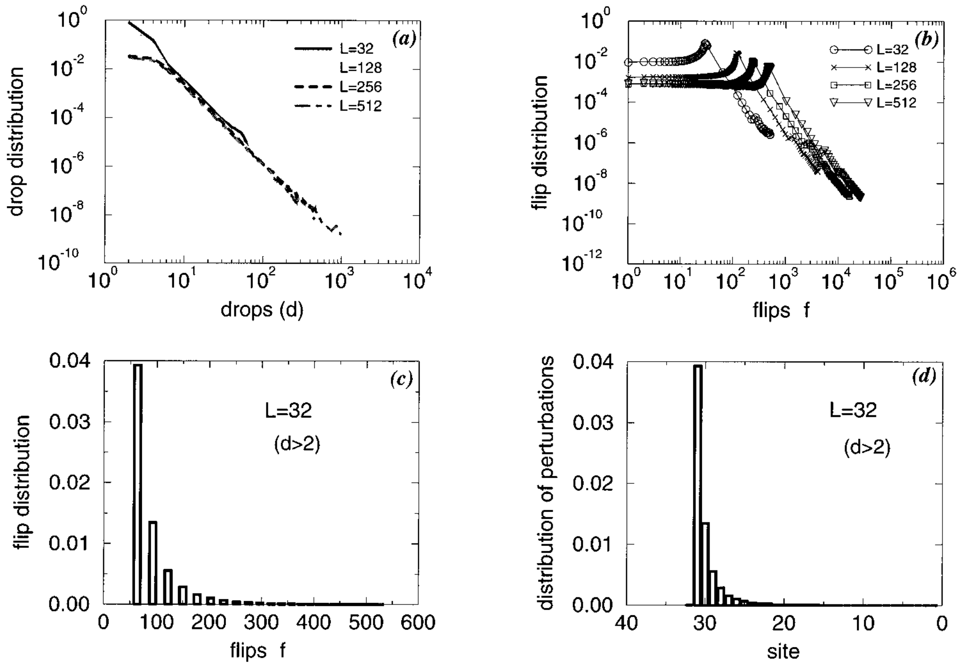

FIG. 3. Statistical analysis of the LLL model with $n_p=1$, $n=2$, and $S_c=2$. (a) Drop distribution for different lattice sizes; (b) Flip distribution for different lattice sizes; (c) Histogram of the flip distribution for $L=32$; (d) Histogram of the distribution of perturbations by site for $L=32$.

forward avalanches are triggered. The possibility of backward avalanches comes from the difference between the number of grains that perturb a site i and the number of grains that topple from site k to site $k-1$. For this reason, there are many possibilities of configurations that correspond to statistically stationary states [for example, the configuration shown in Fig. 2(b)]. As the avalanches are limited by the traps, on a finite lattice it is possible to know beforehand which is the cluster of grains involved in that particular avalanche [see Fig. 2(a)]. In this sense, we say that the traps define the size of the avalanches.

A very simple model, called the ''trough'' or 01 model, where a site i is either occupied by a trap ($\tau=1$) or empty ($\tau=0$) has been proposed by Carlson *et al.* [11] to study the relevance of traps in the behavior of one-dimensional sandpile models. In this trough model, the configurations are reduced to the birth, death, and coalescence of traps. Depending on the death and birth rates, the density of traps behave as a power law with the lattice size. In the LL model, Krug [13] identified three intrinsic length scales which depend on the size of the lattice: (i) the average distance between traps (λ), (ii) the average size of edge events ($d\neq 0$), and (iii) the average size of bulk events ($d=0$). With periodic boundary conditions, Chhabra and collaborators [8] obtained the power laws of two intrinsic length scales, $\lambda \approx L^{1/3}$, and $\xi \approx L^{2/3}$, where ξ is a coherence length.

The backward avalanches and the traps are responsible for the nontrivial behavior of the LL model. The traps explain the existence of intrinsic length scales other than the system size, and the backward avalanches give rise to flip numbers bigger than the lattice size. The analysis of the data, for both drop and flip distributions [see Figs. 2(c) and 2(d)], indicates that a multiscaling fitting seems to give better results than a simple finite-size scaling analysis [7]. Krug [13] has suggested that this multiscaling behavior is related to the existence of two different moments of the distribution of λ, which scale differently with the lattice size.

In the LL model there is a complex behavior which cannot be observed in the BTW1 model. However, we will see that other one-dimensional models are associated with an even higher ''degree of complexity.''

TABLE III. Statistics of perturbations by site and flip distribution ($L=32$ and $d>2$). Note that the distribution of perturbations at site i is the same as the flip distribution of the number of flips $\Sigma_{j=i}^{L} j$.

Site	Distribution of perturbations	Number of flips	Flip distribution
1	0.000 002 3	$63=32+31$	0.039 295 3
2	0.000 006 8	$93=32+31+30$	0.013 404 4
3	0.000 006 8	$122=32+31+30+29$	0.005 545 3
...	...	...	...
...	...	...	...
...	...	...	...
29	0.005 545 3	$525=32+\cdots+3$	0.000 006 8
30	0.013 404 4	$527=32+\cdots+2$	0.000 006 8
31	0.039 295 3	$528=32+\cdots+1$	0.000 002 3

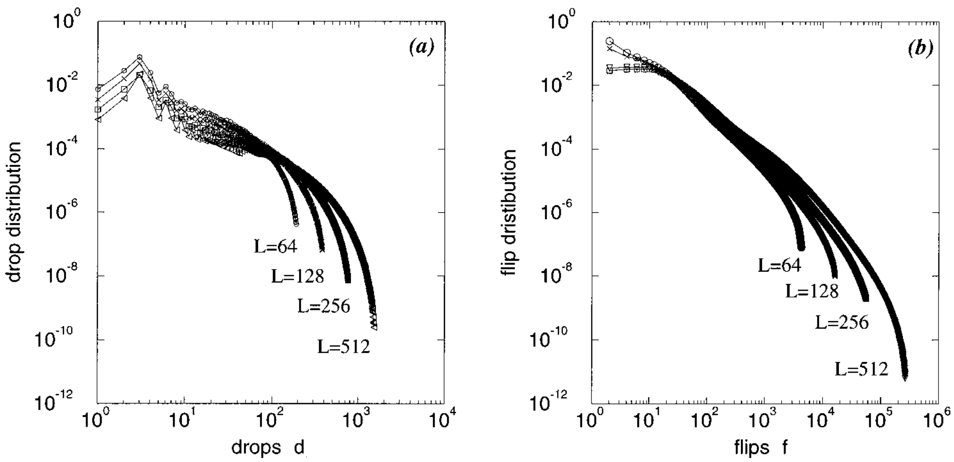

FIG. 4. Statistical analysis of the NLL model with $n_p=1$, $n=2$, and $S_c=2$. (a) Drop distribution for different lattice sizes; (b) Flip distribution for different lattice sizes.

B. The LLL model

Chhabra and collaborators [8] detected a power law in the drop distribution of the LLL model. Besides simulating again this drop distribution [see Fig. 3(a)], we performed some simulations for the flip distribution, which is supposed to gauge the avalanche size more precisely. We have found the superposition of two distinct situations [see Fig. 3(b)]: (i) For $f<L$ $(d=2)$, we reproduce the trivial behavior of the BTW1 model, except for $f\approx L$, in which case the distribution of flips increases because the site L is never forbidden; (ii) For $f>L$ $(d>2)$, we observe a power law. However, instead of being associated with SOC, as claimed by some authors [15], this power law is a manifestation of the asymmetry of the perturbation rule. It just reveals the inhomogeneity in the way the system is perturbed, and cannot be associated with a complex behavior. We remark that the perturbation is forced to be inhomogenous because the density of allowed perturbation sites decays as $1/d$ with the distance d from the closed boundary.

As shown in Figs. 3(c) and 3(d), and in Table III, there is an identification between the statistics of perturbations for $d>2$ and the distribution of flips. For each site k, the ratio between the number of perturbations and the number of avalanches is exactly the same as the flip distribution associated with the number of flips $f=L+(L-1)+\cdots+k$. These results indicate that the closed boundary, together with the restriction imposed in the way the system is perturbed, produces an inhomogeneous distribution of perturbations which is entirely responsible for the power law in the flip distribution. This mechanism is rather simple: for $d>2$, as we can observe in Tables I and II, there is an annihilation reaction, except at $i=L$. In a stationary state, the probability of a backward avalanche is bigger near the closed edge than in the bulk of the lattice. As suggested by different investigations [8,12], the boundary conditions play an essential role in this case. Also, to a certain extent, the behavior is similar to the case of the trivial BTW1 model, as the distribution of flips is directly related to the statistics of the perturbations,

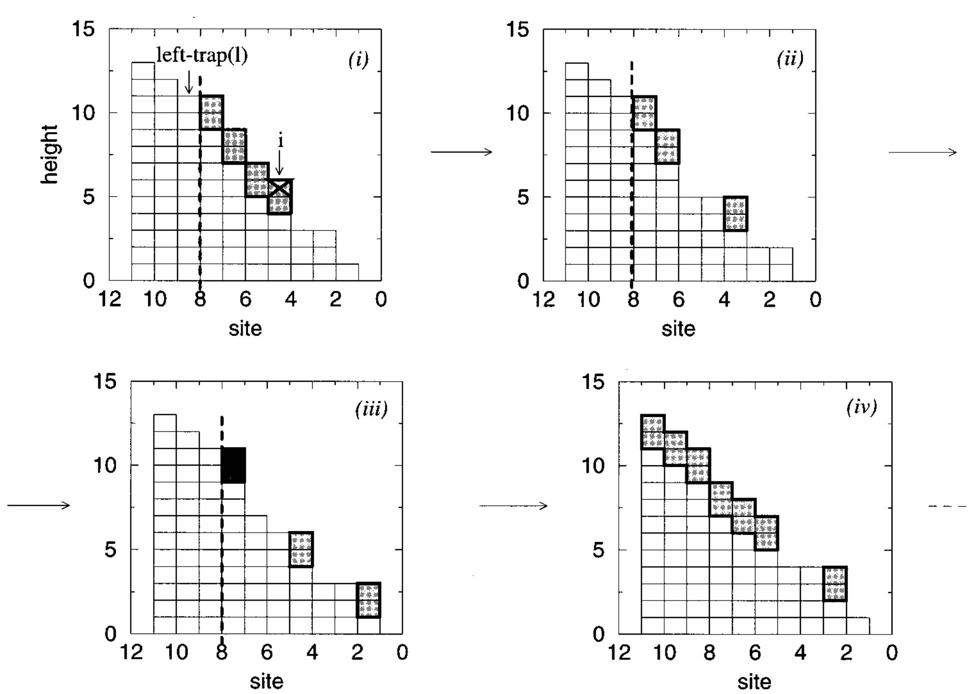

FIG. 5. Example of configuration of the NLL model with $n_p=1$, $n=2$, and $S_c=2$, where new branchings are formed and the avalanche circumvents the ''left trap.'' The left trap, for the first branching, is the site $l>i$, with $\epsilon_l<0$, where i is the perturbed site. The stages ii, iii, and iv are intermediate steps of the evolution. At stage (iii), the new branching at site $l-1$ is reponsible for avoiding the presumed left trap. The intermediate steps are generated to allow the simultaneous updating of all sites k, such that $S_k>S_c$.

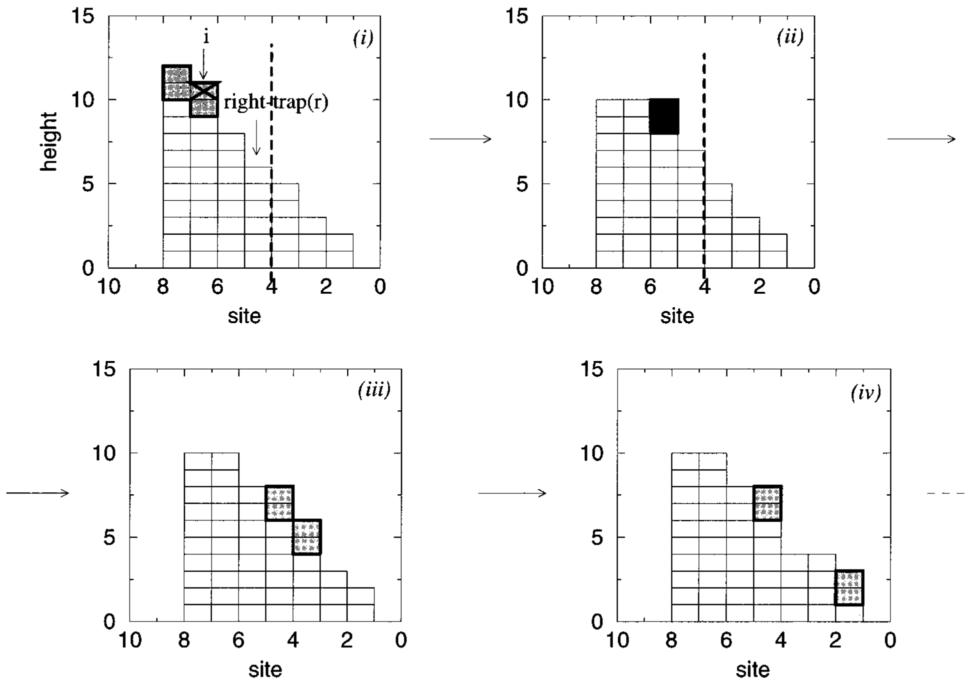

FIG. 6. Example of configuration of the NLL model with $n_p=1$, $n=2$, and $S_c=2$, where new branchings are formed and the avalanche circumvents the ''right trap.'' The right trap, for the first branching, is the site $r=i-p$, where p is an even number, with $\epsilon_r<1$. At stage (ii), the new branching originated at site $r+1$ is responsible for avoiding the presumed right trap. The intermediate steps are generated to allow the simultaneous updating of all sites k, such that $S_k>S_c$.

and the avalanches do not stop before reaching the open edge.

Due to the absence of traps, and the possibility of backward avalanches, for each perturbed site of the LLL model, there are only two possibilities for the distributions of drops and flips. As shown in Table I, one possibility is associated with forward avalanches $(d=2)$, while the other is related to backward avalanches $(d>2)$. The avalanche size can thus be determined univocally from the knowledge of the values of ϵ_i and ϵ_{i+1}, where i is the location of the perturbed site. For each perturbed site, the numbers of drops and flips are determined, as in the trivial BTW1 model. We then conclude that the scale invariance of the LLL model is produced *a priori* by the way the model is defined. It is not reasonable to consider this behavior as complex, at least in the same sense as we refer to a complex behavior in the original two-dimensional model introduced by Bak, Tang, and Wiesenfeld [1]. This is an example of a situation in which a power law is not a manifestation of a complex behavior.

C. The NLL model

The nonlocal limited (NLL) model seems to be the first one-dimensional system to exhibit a complex behavior similar to the two-dimensional models. The simulations for the NLL model show scale invariance in both the drop [see Fig.4(a)] and the flip [see Fig.4(b)] distributions. The number of drops assumes odd and even values due to the dynamical rules of the cellular automaton. As in the nontrivial two-dimensional models, we observed simple scaling in the analysis of the flips, $\rho(f,L)\sim f^{-\delta}$, for small f.

In our simple example $(n_p=1,\ n=2,\ \text{and}\ S_c=2)$, with $\epsilon_i=1$, an avalanche is triggered. Now it is not so easy to know *a priori* where the avalanche stops and what clusters of grains move along the lattice [see Fig. 5(a) and Fig. 5(b)]. The simple motion of grains can produce new branchings (defined by the cluster of grains that move along the lattice) which enhance the avalanche sizes. A presumed trap to the avalanche of the first branching can be transposed by another branching originated after the avalanche has been triggered as we can see in stages (iii) and (iv) of Fig. 5, and stages (ii) and (iii) of Fig. 6). Unlike in the case of the LL model, the dynamics of this model presents avalanches generated by many branchings which are able to circumvent the ''apparent'' traps. It is thus impossible to define the trapping sites, in the sense used before (sites which stop the avalanche), in terms of the values of the variables $\{\epsilon_k\}$ when the system is perturbed. The consequence of this mechanism is the enhancement of the bigger avalanches and the absence of intrinsic length scales in addition to the lattice size. For a finite lattice, it is not possible to define an average distance of traps since, in general, we do not even characterize a trap. In other words, the analysis of configurations of the lattice is not sufficient to explain the complex behavior.

If we consider only one branching for each avalanche, for example the first, it is possible to define a trapping site. The left trap (see Fig. 5) is a site $l>i$, with $\epsilon_l<0$, and the right trap (see Fig. 6) is at site $r=i-p$, where p is an even number, with $\epsilon_r<1$. Using this idea, we simulated a model forbidding the configurations which circumvent these traps. The results reveal that this restriction is not sufficient to recover the LL model. There is a certain number of bigger avalanches that do not even exist in the LL model. This indicates that the configuration of the lattice does not play an important role in the case of this model.

We have also simulated this model forbidding the configurations originated from the new branchings described above. The system evolves to a configuration where $\epsilon_k=1$, for all sites, if any perturbation is forbidden (this is the statistically stationary state of the trivial BTW1 model with $n_p=n=2$). We have another evidence that the branching mechanisms are responsible for the nontrivial behavior of the NLL model.

The NLL model is thus more similar (than the LL model) to the nontrivial two-dimensional models. At each motion of grains, at least two sites are independently perturbed as in the two-dimensional models. Also, the simple scaling observed in the flip distribution is a manifestation of this similarity. This kind of behavior certainly corresponds to a higher ''degree of complexity'' than in the case of the LL model.

IV. CONCLUSIONS

The presence of scale invariance in one-dimensional sandpile models does not necessarily imply a complex behavior. To exhibit SOC, at least in the sense originally proposed by Bak, Tang, and Wiesenfeld, it is required that the existence of branching mechanisms that provide the possibility of an occurrence of avalanches of all sizes, as in the NLL model. It is even possible to distinguish two levels of complexity, for the LL and NLL models. Considering a finite lattice, the LL model presents intrinsic length scales related to the density of traps on the lattice. In the NLL model, there are no extra length scales, even on a finite lattice, due to the appearance of new branchings during the relaxation process (which leads to the recovery of the simple scaling behavior observed in the two-dimensional models).

In conclusion, we have raised some questions about the connections between SOC, SI, and complexity. The presence of power laws in the statistical analysis of the systems that we have considered should not be taken as a synonymous of complex behavior. Also, it should be pointed that scaling invariance can be a manifestation of different types of behavior, including some rather trivial situations, as in the LLL model, where the boundary conditions play a determinant role.

Finally, we wish to emphasize the coincidence between a more complex behavior, as in the NLL and the two-dimensional BTW models, and the existence of a simple scaling law. Although we have not investigated this point in detail, we suggest that the character of the scaling could be used to distinghish the degrees of complexity of systems as the LL and the NLL models.

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